Kuwait University
Department of Mathematics & Computer Science

Math 101 — January 12, 2003
Calculus I — Final Examination — Time: 2 hours

Calculators, mobile phones and pagers are not allowed during this exam.
Each question is worth 5 points.

1. Find the limit if it exists:

$$(a)\quad \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} \qquad\qquad (b)\quad \lim_{x\to 0} x^4 \sin\frac{1}{\sqrt[3]{x}}$$

2. Use the definition of the derivative to show that the following function is differentiable at $x=\frac{1}{2}$:

$$f(x)=\begin{cases}\cos(2x-1)+4x^2-2x &, \quad \text{if } x\le \frac{1}{2}\\[2mm] 2-\dfrac{1}{2x} &, \quad \text{if } x>\frac{1}{2}\end{cases}$$

3. Let $f(x)=x\cos x$. Use Rolle's theorem to show that $f''(c)=0$ for some $c\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

4. Show that the following function is constant on $(0,\infty)$:

$$F(x)=\int_x^{3x}\frac{1}{t}\,dt$$

5. Evaluate the integral $\displaystyle\int_{-2}^{2}\left[\cos(\pi x)-\frac{x^3}{\sqrt{1+x^4}}\right]dx$

6. Let $f(x)=\dfrac{x}{\sqrt{x^2+9}}$. What is the average value of $f(x)$ on $[0,4]$? Find a number z that satisfies the conclusion of the mean value theorem for definite integrals.

7. Find the area of the region enclosed by the graph of $y=x^2$ and the lines $x=2$ and $y=0$

8. Find the resulting volume when the region bounded by the graphs of $x=(y-1)^2$ and $x=y+1$ is revolved about the line $y=3$.

1. a) $\lim_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \dfrac{d}{dx}(x^3) = 3x^2.$

or $\lim_{h\to 0} \dfrac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \dfrac{(\cancel{x+h-x})((x+h)^2+(x+h)x+x^2)}{\cancel{h}} = 3x^2$

b) $\lim_{x\to 0} x^4 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$ that by using the sandwich theorem.

2. $\lim_{x\to 1/2} \dfrac{f(x) - f(1/2)}{x - 1/2} \Rightarrow \lim_{x\to 1/2^-} \dfrac{\cos(2x-1) + 4x^2 - 2x - 1}{x - 1/2} = 2\lim_{x\to \frac{1}{2}^-} \dfrac{\cdots}{2x-1}$

$= 2\lim_{x\to 1/2^-} \dfrac{\cos(2x-1) - 1 + 2x(2x-1)}{2x-1} = 2,$

$\lim_{x\to 1/2^+} \dfrac{2 - 1/(2x) - 1}{x - 1/2}\cdot\dfrac{2x}{2x} = \lim_{x\to 1/2^+} \dfrac{2x-1}{2x(\cancel{x-1/2})} = 2. \Rightarrow f'(1/2) = 2.$

3. $f'(x) = \cos x - x\sin x$ satisfies the hypothesis of Rolle's theorem on $[-\pi/2, \pi/2]$. $f'(-\pi/2) = -\pi/2 = f(\pi/2)$. Hence, there exists $c \in (-\pi/2, \pi/2)$ such that $f''(c) = 0$.

4. $F(x) = \int_x^{3x} \frac{1}{t}\,dt$, $F'(x) = \frac{1}{3x}\cdot 3 - \frac{1}{x} = 0$

so $F(x)$ is a constant function.

5. $\int_{-2}^{2} \underbrace{\cos(\pi x)}_{\text{even}} - \underbrace{\frac{x^3}{\sqrt{1+x^4}}}_{\text{odd}}\,dx = \int_{-2}^{2} \cos(\pi x)\,dx$ | put $t = \pi x$, $dt = \pi\,dx$; $x = 0 \Rightarrow t = 0$; $x = 2 \Rightarrow t = 2\pi$

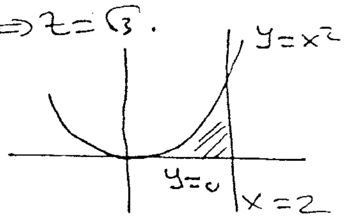
$= 2\int_0^2 \cos(\pi x)\,dx = \frac{2}{\pi}\int_0^{2\pi} \cos(t)\,dt = 0.$

6. $f_{av} = \frac{1}{4}\int_0^4 \frac{x}{\sqrt{x^2+9}}\,dx = \frac{1}{4}\cdot\frac{1}{2}\int_0^4 \frac{2x}{\sqrt{x^2+9}}\,dx$ $\quad t = (x^2+9)$, $dt = 2x\,dx$

$= \frac{1}{8}\int_9^{25} \frac{dt}{\sqrt{t}} = \frac{1}{4}\sqrt{t}\Big|_9^{25} = 1/4(5-3) = 1/2$ $\quad x=0 \Rightarrow t=9$, $x=4 \Rightarrow t=25$

$f(z) = f_{av} \Rightarrow z/\sqrt{z^2+9} = 1/2 \Rightarrow 2z = \sqrt{z^2+9} \Rightarrow z = \sqrt{3}.$

7. $A = \int_0^2 x^2\,dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}.$



8. $V = \int_0^3 2\pi(3-y)\left((y+1) - (y-1)^2\right)dy$

$= ----$